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COMMENT

# The equivalence of $m$ quasiparticles to one particle in the Laughlin state

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**Abstract.** On the basis of complex variable theory, we present an alternative demonstration of the equivalence of $m$ quasiparticles to one true particle in the Laughlin state.

Following a fascinating paper by Laughlin [1], numerous theoretical works have shown that the ground state of the fractional quantum Hall effect (FQHE) is well described by Laughlin's many-electron variational wavefunction [2]. Recently, several authors have discussed the off-diagonal long-range order (ODLRO) in the FQHE ground state [3-5]. The basic idea in their arguments is the equivalence of $m$ quasiparticles to one true particle in the Laughlin state $\nu = 1/m$. As first noted by Halperin [6], this property is specific to the Laughlin states for $\nu = 1/m$. Although this equivalence has been suggested tacitly in Laughlin's gauge argument [1, 2], his argument may have a certain intuitive appeal, however. Arovas *et al* [7] have offered an explanation by way of a Berry phase calculation. In this comment, we show that this equivalence can also be derived from complex variable theory.

We start by quoting some mathematical theorems, which will be used in the following discussion. Most of them can be found in [8].

Let $P(z)$ be a complex polynomial with the following form:

$$P(z) = \prod_{j=1}^{N} (z - \alpha_j)^m \qquad m = \text{positive integer} \tag{1}$$

then

$$P(\alpha_j) = 0 \qquad \text{for } j = 1, 2, \ldots, N. \tag{2}$$

where the $\alpha_j$ are the $m$-fold zero points of $P(z)$.

*Theorem 1.* All zero points of $P'(z)$ are contained in the zero-point polygon of $P(z)$†.

*Theorem 2.* All zero points of $P(z)$ are contained in the zero-point polygon of $zP(z)$.

Theorem 1 established the connection between the zeros of $P(z)$ and its derivative $P'(z)$, which can be considered as a counterpart of the well known Rolle theorem in the complex variable domain. Theorem 2 can be proved readily by using theorem 1.

† We term a minimal closed convex curve in the $z$ plane, which contains all zero points of $P(z)$ inside and on itself, a zero-point polygon of $P(z)$.

*Theorem 3.* Let $C$ be a contour in the $z$ plane‡. Suppose that $P(z)$ is analytic everywhere inside and on $C$, and $P(z)$ has $N$ zeros, the $j$th zero being at $a_j$ and is of order $n_j$, in the interior of $C$; furthermore, $P(z)$ is non-zero on $C$; then

$$(1/2\pi\mathrm{i}) \oint_C (P'(z)/P(z))\,\mathrm{d}z = (1/2\pi)\Delta_C \arg P(z) = \sum_{j=1}^{N} n_j \tag{3}$$

where $\Delta_C \arg P(z)$ denotes the variation of arg $P(z)$ (the argument of $P(z)$) after the variable $z$ traverses around along $C$ in a counterclockwise direction.

Theorem 3 is a special case of the argument principle.

*Theorem 4.* The number of zeros of $P'(z)$ inside $C$ is less than the number of zeros of $P(z)$ in the same region by unity.

*Theorem 5.* The number of zeros of $zP(z)$ inside $C$ exceeds the number of zeros of $P(z)$ in the same region by unity.

Theorems 4 and 5 are the trivial consequences of Macdonald's result [8].

We next show how the equivalence of $m$ quasiparticles to one true particle in the Laughlin state can be obtained by virtue of the theorems quoted above.

Let Laughlin's wavefunction be the form

$$\psi_m \sim \prod_{i<j} (z_i - z_j)^m \qquad z_i = (x_i, y_i) \qquad m = \text{odd} \tag{4}$$

where, for convenience, the exponential factor $\exp(-\Sigma_k |z_k|^2/4l_0^2)$ has been omitted. Likewise, let Laughlin's wavefunction for a quasi-electron and quasi-hole centred at $\xi = (\xi_x, \xi_y)$ and $\eta = (\eta_x, \eta_y)$ be the following form, respectively:

$$\psi_m^{\mathrm{e}} \sim \prod_i (2l_0^2(\partial/\partial z_i) - \xi^*) \prod_{i<j} (z_i - z_j)^m \qquad \text{quasi-electron} \tag{5}$$

$$\psi_m^{\mathrm{h}} \sim \prod_i (z_i - \eta) \prod_{i<j} (z_i - z_j)^m \qquad \text{quasi-hole.} \tag{6}$$

Put $l_0 = 1$, $\eta = \xi^* = 0$ and let $P(z_i) = \prod_j (z_i - z_j)^m$; we have

$$\psi_m^{\mathrm{e}} \sim \prod_i (\partial/\partial z_i) P(z_i) \tag{7}$$

$$\psi_m^{\mathrm{h}} \sim \prod_i z_i P(z_i). \tag{8}$$

Firstly, one should note that (i) in the presence of a magnetic field, the wavefunction is no longer real but complex; (ii) in Laughlin's wavefunction (4), $\psi_m \to 0$ when $z_i \to z_j$ due to Pauli's principle, i.e. the zero points of Laughlin's wavefunction do exist in the electron position $z_j$ [9].

(A) Let the zero point $\alpha_j$ of $P(z)$ (equation (1)) correspond to the electron position $z_j$ of Laughlin's wavefunction (4); we have the following correspondence:

the distribution of electron positions of $\psi_m(\psi_m^{\mathrm{e}}, \psi_m^{\mathrm{h}})$

$\leftrightarrow$ the distribution of zero points of $P(z)$ $(P'(z), zP(z))$.

On the basis of theorems 1 and 2, we can understand qualitatively that the creation of Laughlin's quasi-electron state $\psi_m^{\mathrm{e}}$ (or quasi-hole state $\psi_m^{\mathrm{h}}$) from $\psi_m$ is a redistributing

‡ Throughout this paper, we assume that the contour $C$ is large enough so that all zero points of a complex polynomial are included in it.

process of electron position $z_j$ in the $z$ plane, i.e. the creation of Laughlin's quasi-electron (quasi-hole) state $\psi_m^{\mathrm{e}}$ ($\psi_m^{\mathrm{h}}$) from $\psi_m$ is a redistributing process of electron position $z_j$ toward concentration of (separation from) each other.

(B) Applying theorem 3 to Laughlin's wavefunction (4), we conclude that

> if we move an electron around a contour $C$, enclosing other electrons in the system, then the phase of the wavefunction changes by an amount $2\pi mN$ ($\because n_j = m$ in (3))

where $N$ is the total number of electrons included in the $C$. This conclusion is consistent with Halperin's result [6].

If we term a zero point of Laughlin's wavefunction (4), which contributes to the phase change by $2\pi$, a 'vortex zero', then the above conclusion can be restated as follows:

> one electron is equivalent to $m$ vortices.

Further, if we consider the vortex a quasiparticle, then $m$ quasiparticles are equivalent to one true particle. In particular, $m$ quasi-electrons (quasi-holes) are equivalent to one electron (hole).

(C) On the basis of theorems 4 and 5, and the discussion in (B), we have the following quantitative conclusion:

> the number of zeros of $\psi_m^{\mathrm{e}}$ inside $C$ is less than the number of zeros of $\psi_m$ in the same region by unity, i.e. create a quasi-electron;
>
> the number of zeros of $\psi_m^{\mathrm{h}}$ inside $C$ exceeds the number of zeros of $\psi_m$ in the same region by unity, i.e. create a quasi-hole.

Finally, we discuss the correctness of the above conclusion from a physical point of view. Following Halperin [6], we investigate a phase change of an $N$-electron wavefunction when the positions of all but one electron are fixed and the remaining electron moves around a contour $C$, which encloses an area $S$. The magnetic field $B$ requires a phase change of $\Delta\phi = 2\pi\phi/\phi_0 = S/l_0^2$, where $\phi = BS$, $\phi_0 = hc/e$ and $l_0^2 = \hbar c/eB$. For the Laughlin state (4), the number of electrons inside $C$ is $N = S/2\pi l_0^2 m$. Since around each electron the phase of the wavefunction changes by $2\pi m$, then the total phase change is $S/l_0^2$, just the amount required by the magnetic field $B$. Namely, the magnetic field requires $1/2\pi l_0^2$ vortices per unit area, and in the Laughlin state (4) $m$ vortices are attached to each electron, hence the requirement is satisfied.

Let the system expand a little from $\nu = 1/m$; we have to introduce quasi-holes to the Laughlin state (4). Since the decrease in the electron density due to the system expansion lowers the vortex attached to the electrons, the system needs more vortices to fulfil the requirement of the magnetic field. For instance, the area of the system expands by $2\pi l_0^2$ and one more vortex is necessary. Because of the Fermi statistics, an even number of vortices cannot be attached to an electron; the extra vortex enters the system as a free vortex. This vortex is a so-called quasi-hole.

When we compress the system a little from $\nu = 1/m$, we can produce a quasi-electron. Since we presently consider only the lowest Landau level, we cannot introduce an anti-vortex in the system (because around the anti-vortex the angular momentum of an electron is $-\hbar$, it is only possible when the electron is in the higher Landau level). For the same reason as mentioned above, we remove two vortices from an electron and discard one vortex while the other is left as a free vortex in the system. This vortex is a so-called quasi-electron.

In closing, we comprehensively describe our result as follows.

(1) The creation of Laughlin's quasi-electron state $\psi_m^{\rm e}$ from $\psi_m$ is a redistributing process of $N$ electrons in the system toward concentration of each other; the result is $N$ electrons plus one quasi-electron.

(2) The creation of Laughlin's quasi-hole state $\psi_m^{\rm h}$ from $\psi_m$ is a redistributing process of $N$ electrons in the system toward separation from each other; the result is $N$ electrons plus one quasi-hole.

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